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Natural resources and environmental economics, TSE

Tutorial 5a - Exhausible Resources - Market Economies
SOLUTIONS

Solution to Exercise 1c: A Monopoly with linear inverse demand

The linear demand is given by the form $p_t(s_t) = a - bs_t$ and its (inverse) demand elasticicy can be computed as $\varepsilon_{p,s} = \left|\frac{-bs_t}{a-bs_t}\right| = \frac{1}{\frac{a}{bs_t}-1}$. Note that the demand is zero at the price larger than a and the maximum quantity that will be consumed is a/b. The elasticity is zero for small s_t and tends towards infinity for $s_t \to s_{\max} = a/b$. Note that here we are considering the INVERSE demand function which is increasing in s. This implies that the demand is less elastic for small values of s_t (as the inverse demand is more elastic).

Proceeding as usual we can directly write the Hamiltonian as

$$H = (a - bs_t)s_t e^{-rt} + \lambda_t(-s_t)$$

which yields the FOCs

$$\frac{\partial H}{\partial s_t} = (a - 2bs_t)e^{-rt} - \lambda_t \stackrel{!}{=} 0 \text{ and } \frac{\partial H}{\partial S_t} = 0 \stackrel{!}{=} -\dot{\lambda}_t.$$

It is easy to see that in this case with a linear demand, the resource will be depleted in finite time. Otherwise, at some point the (always increasing) price will be higher than a and demand would drop to zero so it must be that at some point T the resource is exhausted. Using both first order conditions, we can solve for s_t depending on λ as

$$s_t = \frac{a}{2b} - \frac{\lambda}{2b}e^{rt}.$$

At least here you should be able to recognize that depletion occurs in finite time since this term will be negative otherwise (for large t it is always negative but you cannot have negative resource consumption).

As in the previous problem set, we can compute the time at which the resource is depleted as the value t for which $s_t = 0$. This yields $T = \log\left(\frac{a}{\lambda}\right)^{1/r} = \frac{\log a - \log \lambda}{r}$

Again now we use the resource constraint which in this case has the finite time horizon T (its value we have computed before!) so that

$$\hat{S}_0 = \int_0^T s_t dt = \left[\frac{a}{2b} - \frac{\lambda}{2b}e^{rt}\right]_0^T = \frac{a}{2b}T - \frac{\lambda}{2br}e^{rT} + \frac{\lambda}{2br}$$

and using the value of T we finally get $\hat{S}_0 = \frac{a}{2b}\log\left(\frac{a}{\lambda}\right)^{1/r} - \frac{\lambda}{2br}(1 - \frac{a}{\lambda})$. This admits no analytical solution for λ. However, one can e.g. actually show that using the Implicit Function Theorem that e.g. the shadow value of the resource λ decreases if the initially available amount of S_0 increases.

However, in this case you are NOT supposed to show this as it is too complicated (and actually boring to do so, at least analytically)!

The important message is the intuition of the results "in the middle of the derivations", e.g., that the depletion time $T = \log\left(\frac{a}{\lambda}\right)^{1/r}$ decreases if the interest rate increases.

Solution to Exercise 2: Extraction in the competitive case (no extraction costs)

Since under perfect competition, maximizing profits is eauivalent with maximizing consumers' surplus, the problem of the competitive firm is

$$\max_{s_t} \int_0^\infty \left[\int_0^{s_t} p(x)dx\right] e^{-rt}dt \quad \text{s.t.} \quad \hat{S} = \int_0^\infty s_t dt, \dot{S}_t = -s_t.$$

The Hamiltonian for this problem is just $H = \left[\int_0^{s_t} p(x)dx\right] e^{-\rho t} + \lambda_t(-s_t)$. And we find for the first order conditions

$$\begin{aligned} \frac{\partial H}{\partial s_t} &= p(s_t)e^{-\rho t} - \lambda_t \stackrel{!}{=} 0 \\ \frac{\partial H}{\partial S_t} &= 0 \stackrel{!}{=} -\dot{\lambda}_t \end{aligned}$$

(The derivative of consumer surplus as simply $p(s_t)$ can be actually derived using the Leibniz rule). From the second condition, we get that λ_t is constant over time so that by rearranging the first condition we have immediately the condition $p(s_t) = \lambda e^{\rho t}$. So logdifferentiating this expression gives us the Hotelling rule with the interest rate as $\frac{\dot{p}}{p} = r$. Note that extraction in the competitive case is therefore optimal if the interest rate is equal to the social rate of time preference ρ.

Solution to Exercise 3: A simple optimal growth model

a) The Social Planner is maximizing discounted utility over c_t: $\max_{c_t} \int_0^\infty (\ln c_t)e^{-\rho t}dt$ s.t. $\dot{k}_t = k_t^\alpha - c_t$. The state variable is k_t while the control variable is c_t.

b) The Hamiltonian reads therefore $H = (\ln c_t)e^{-\rho t} + \lambda_t(k_t^\alpha - c_t)$. The first order conditions are

$$\begin{aligned} \frac{\partial H}{\partial c_t} &= \frac{1}{c_t}e^{-\rho t} - \lambda_t \stackrel{!}{=} 0 \\ \frac{\partial H}{\partial k_t} &= \lambda_t \alpha k_t^{\alpha-1} \stackrel{!}{=} -\dot{\lambda}_t. \end{aligned}$$

logdifferentiating the first condition, we get $\frac{\dot{c}_t}{c_t} + \rho = -\frac{\dot{\lambda}_t}{\lambda_t}$. From the second condition we get another relation for $-\frac{\dot{\lambda}_t}{\lambda_t}$ simply by dividing by λ_t, namely $\alpha k_t^{\alpha-1} = -\frac{\dot{\lambda}_t}{\lambda_t}$. By equalizing these two expressions, we can get rid completely of the costate variable λ_t and get immediately the condition

$$\frac{\dot{c}_t}{c_t} = \alpha k_t^{\alpha-1} - \rho.$$